4. Test on population variance when population mean is known.

To test $H_0: \sigma^2 = \sigma_0^2$
against $H_1: \sigma^2 > \sigma_0^2$
$\sigma^2 < \sigma_0^2$
$\sigma^2 \neq \sigma_0^2$

Let $X_1, X_2, \ldots, X_n \sim N(\mu, \sigma^2)$
μ known
σ^2 unknown.

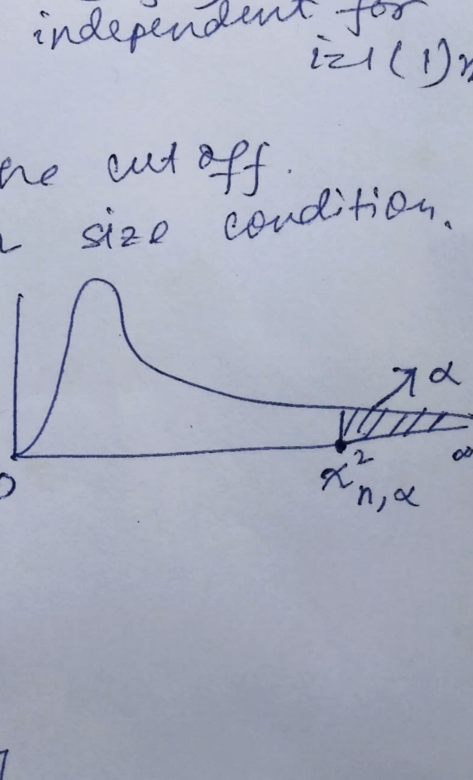
The unbiased estimate of σ^2 is
$$s_0^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2.$$

So the test statistic will be based on s_0^2.
Intuitively, a large value of $\frac{s_0^2}{\sigma^2}$ will lead us to the rejection of hypothesis, H_0.

Now, $X_i \sim N(\mu, \sigma^2)$
$$\Rightarrow \left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_{(1)}$$
$$\Rightarrow (X_i-\mu)^2 \sim \sigma^2 \cdot \chi^2_{(1)}$$
, $X_1, X_2, \ldots, X_n$ are i.i.d. r.v.
$$\Rightarrow \frac{1}{\sigma^2}\Sigma(X_i-\mu)^2 \sim \chi^2_{(n)}$$
so $\left(\frac{X_i-\mu}{\sigma}\right)^2$'s are all independent for $i=1(1)n$
$$\Rightarrow \frac{n s_0^2}{\sigma^2} \sim \chi^2_{(n)}.$$

We reject hypothesis if
$\chi^2_{calc} = \frac{n s_0^2}{\sigma_0^2} > c$ where the cut off point c will be calculated from size condition.

$$P_{H_0}\left[\frac{n s_0^2}{\sigma^2} > c\right] = \alpha.$$
$$\Rightarrow c = \chi^2_{n,\alpha}$$



$\Rightarrow$ reject H_0 if
$$\chi^2_{cal} = \frac{n s_0^2}{\sigma_0^2} > \chi^2_{n,\alpha}$$
$$\boxed{\Rightarrow s_0^2 > \frac{\sigma_0^2}{n}\chi^2_{n,\alpha}}$$

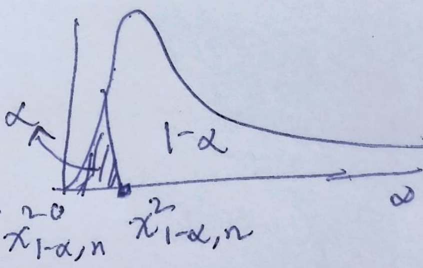
$$\text{Power} = P\left[\frac{n s_0^2}{\sigma_0^2} > \chi^2_{n,\alpha} \,/\, H_1 \text{ true}\right]$$
$$= P\left[\frac{n s_0^2}{\sigma^2}\cdot\frac{\sigma^2}{\sigma_0^2} > \chi^2_{n,\alpha}\right]$$
$$= P\left[\chi^2_n > \frac{\sigma_0^2}{\sigma^2}\chi^2_{n,\alpha}\right]$$

Now, if $H_1: \sigma^2 < \sigma_0^2$

Test statistic $\frac{ns_0^2}{\sigma^2} \sim \chi^2_n$

But the rejection will be towards the left tail. Lesser the value of s_0^2 as compared with σ^2 under H_0, lesser will be the ratio, value of test statistic, thereby rejecting H_0.

$\boxed{\text{Remember}}$ χ^2 is a positively skewed (not symmetric) distribution, $0 < \chi^2 < \infty$.



So we reject H_0 if

$$\chi^2 = \chi^2_{cal} = \frac{ns_0^2}{\sigma_0^2} < \chi^2_{1-\alpha,n}$$

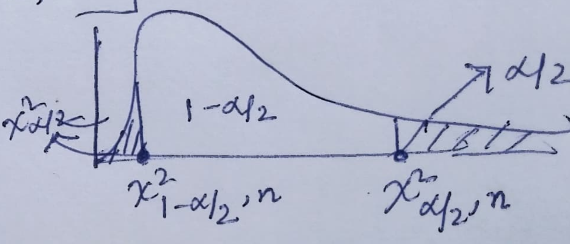
$$\Rightarrow s_0^2 < \frac{\sigma_0^2}{n}\chi^2_{1-\alpha,n}$$

$$\text{Power} = Pr\left[\frac{ns_0^2}{\sigma_0^2} < \chi^2_{1-\alpha,n} \Big/ H_1\right]$$

$$= Pr\left[\frac{ns_0^2}{\sigma^2}\cdot\frac{\sigma^2}{\sigma_0^2} < \chi^2_{1-\alpha,n}\right]$$

$$= Pr\left[\chi^2 < \frac{\sigma_0^2}{\sigma^2}\cdot\chi^2_{1-\alpha,n}\right]$$

<u>Case III</u> $H_1: \sigma^2 \neq \sigma_0^2$



We reject H_0 if

$\chi^2_{cal} > \chi^2_{\alpha/2,n}$ or

$\chi^2_{cal} < \chi^2_{1-\alpha/2,n}$.

You can find power accordingly.

Test on σ^2 when μ is <u>not</u> known.

$H_0: \sigma^2 = \sigma_0^2$

$H_1: \sigma^2 > \sigma_0^2$

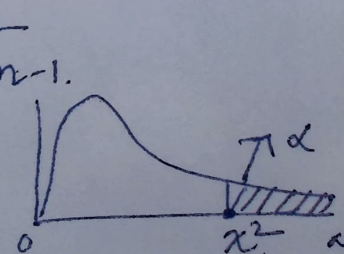
Test statistic will be changed

$$s^2 = \frac{1}{n-1}\Sigma(x_i - \bar{x})^2 = \text{sample variance}$$

is unbiased estimate of σ^2.

So, the test statistic $\frac{(n-1)s^2}{\sigma^2} = \chi^2_{n-1}$.

$$\chi^2_{cal} = \frac{(n-1)s^2}{\sigma_0^2}$$



We reject H_0, if $\chi^2_{cal} > \chi^2_{n-1,\alpha}$

For, $H_1: \sigma^2 < \sigma_0^2$, $\chi^2_{cal} < \chi^2_{1-\alpha,n-1}$.

For $H_1: \sigma^2 \neq \sigma_0^2$, $\chi^2_{cal} > \chi^2_{\alpha/2,n-1}$ or $\chi^2_{cal} < \chi^2_{1-\alpha/2,n-1}$